國立板橋高級中學 100 學年度數學科菩提盃雙週解題《第六回》參考解答

高一：設a,b為實數，且$ab-a-b-1=0$，試求ab的範圍？

【解】令$ab=t$，則$a+b=t-1$，故a,b為$x^2-(t-1)x+t=0$的兩根

判別式$=(t-1)^2-4t\ge 0 \Rightarrow t\ge 3+2\sqrt{2}$ 或 $t\le 3-2\sqrt{2}$

【解】$ab-a-b-1=0 \Rightarrow (a-1)(b-1)=2 \Rightarrow a-1,b-1$兩數同正或同負

(1)當兩數同正時，則$\dfrac{(a-1)+(b-1)}{2}\ge\sqrt{(a-1)(b-1)}=\sqrt{2} \Rightarrow a+b-2\ge 2\sqrt{2}$

故$ab=a+b+1\ge 3+2\sqrt{2}$

(2)當兩數同負時，則$\dfrac{(1-a)+(1-b)}{2}\ge\sqrt{(1-a)(1-b)}=\sqrt{2} \Rightarrow 2-a-b\ge 2\sqrt{2}$

故$ab=a+b+1\le 3-2\sqrt{2}$

【解】$ab-a-b-1=0 \Rightarrow a=\dfrac{b+1}{b-1} \Rightarrow ab=\dfrac{b^2+b}{b-1}=b+2+\dfrac{2}{b-1}$

(1)當$b-1>0$時，則$\dfrac{(b-1)+\dfrac{2}{b-1}}{2}\ge\sqrt{(b-1)\times\dfrac{2}{b-1}}=\sqrt{2} \Rightarrow (b-1)+\dfrac{2}{b-1}\ge 2\sqrt{2}$

故$ab=3+b-1+\dfrac{2}{b-1}\ge 3+2\sqrt{2}$

(2)當$b-1<0$時，則$\dfrac{(1-b)+\dfrac{2}{1-b}}{2}\ge\sqrt{(1-b)\times\dfrac{2}{1-b}}=\sqrt{2} \Rightarrow (1-b)+\dfrac{2}{1-b}\ge 2\sqrt{2}$

故$ab=3-\left[(1-b)+\dfrac{2}{1-b}\right]\le 3-2\sqrt{2}$

【解】由柯西不等式：$(a^2+b^2)(1^2+1^2)\ge(a+b)^2 \Rightarrow \left[(a+b)^2-2ab\right]\times 2\ge(a+b)^2$

$\Rightarrow \left[(ab-1)^2-2ab\right]\times 2\ge(ab-1)^2 \Rightarrow (ab)^2-6ab+1\ge 0$

$\Rightarrow ab\ge 3+2\sqrt{2}$ 或 $ab\le 3-2\sqrt{2}$

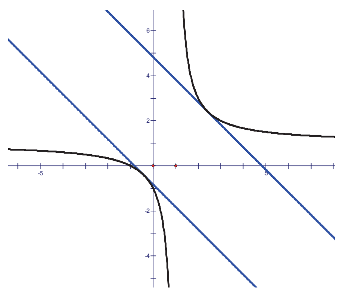
【解】$ab-a-b-1=0 \Rightarrow ab=a+b+1$，$(a-1)(b-1)=2$

令$ab=k$，本題可視為直線$a+b+1=k$與雙曲線$(a-1)(b-1)=2$的相交情形

切點坐標為$(\sqrt{2}+1,\sqrt{2}+1)$或$(1-\sqrt{2},1-\sqrt{2})$

故$k\ge 3+2\sqrt{2}$ 或 $k\le 3-2\sqrt{2}$

即$ab\ge 3+2\sqrt{2}$ 或 $ab\le 3-2\sqrt{2}$



高二：設x,y為實數，且$x^2+y^2-6x-4y+9=0$，試求$\dfrac{y}{x}$的最大值為何？

〚解〛設$\dfrac{y}{x}=m$，將$y=mx$代入$x^2+y^2-6x-4y+9=0$得：

$$x^2+(mx)^2-6x-4mx+9=0 \Rightarrow (m^2+1)x^2-(4m+6)x+9=0$$

判別式$=(4m+6)^2-4\times 9\times(m^2+1)\geq 0 \;\Rightarrow\; 5m^2-12m\leq 0 \;\Rightarrow\; 0\leq m\leq\dfrac{12}{5}$

故$\dfrac{y}{x}$的最大值為$\dfrac{12}{5}$

〚解〛 $x^2+y^2-6x-4y+9=0 \;\Rightarrow\; (x-3)^2+(y-2)^2=2^2$

設$\dfrac{y}{x}=m$，則$y=mx$，本題可視為直線$y=mx$與圓$(x-3)^2+(y-2)^2=2^2$的相切情形

$\dfrac{|3m-2|}{\sqrt{m^2+1}}=2 \;\Rightarrow\; \dfrac{9m^2-12m+4}{m^2+1}=4 \;\Rightarrow\; 5m^2-12m=0 \;\Rightarrow\; m=0$或$\dfrac{12}{5}$

故$\dfrac{y}{x}$的最大值為$\dfrac{12}{5}$

〚解〛設$\dfrac{y}{x}=m$，則$mx-y=0$

由柯西不等式：$\left[(x-3)^2+(y-2)^2\right]\left[m^2+(-1)^2\right]\geq(mx-3m-y+2)^2$

$\Rightarrow\; 4(m^2+1)\geq(3m-2)^2 \;\Rightarrow\; 5m^2-12m\leq 0 \;\Rightarrow\; 0\leq m\leq\dfrac{12}{5}$

〚解〛 $(x-3)^2+(y-2)^2=2^2 \;\Rightarrow\; \begin{cases} x=3+2\cos\theta \\ y=2+2\sin\theta \end{cases} \;\Rightarrow\; \dfrac{y}{x}=\dfrac{2+2\sin\theta}{3+2\cos\theta}$

令$\dfrac{2+2\sin\theta}{3+2\cos\theta}=m \;\Rightarrow\; 2+2\sin\theta=m(3+2\cos\theta)$

$\Rightarrow\; 2-3m=2(m\cos\theta-\sin\theta)=2\sqrt{m^2+1}\cos(\theta+\varphi)\quad\left(\cos\varphi=\dfrac{m}{\sqrt{m^2+1}},\ \sin\varphi=\dfrac{1}{\sqrt{m^2+1}}\right)$

$\Rightarrow\; |2-3m|\leq 2\sqrt{m^2+1} \;\Rightarrow\; 4-12m+9m^2\leq 4m^2+4 \;\Rightarrow\; 5m^2-12m\leq 0 \;\Rightarrow\; 0\leq m\leq\dfrac{12}{5}$