國立板橋高級中學100學年度數學科菩提盃雙週解題《第一回》參考解答

高一：求方程式$(37-5x)^3+(43+2x)^3=(80-3x)^3$的解。

【解】令$a=37-5x,\ b=43+2x$，則原式可改寫為$a^3+b^3=(a+b)^3$

又$(a+b)^3=a^3+b^3+3ab(a+b)\quad \therefore 3ab(a+b)=0$

即$(37-5x)(43+2x)(3x-80)=0\Rightarrow x=\dfrac{37}{5},-\dfrac{43}{2},\dfrac{80}{3}$

【解】令$a=37-5x,\ b=43+2x,\ c=3x-80$，則$a+b+c=0$，且原式可改寫為$a^3+b^3+c^3=0$

又$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

所以$3abc=0\Rightarrow(37-5x)(43+2x)(3x-80)=0\Rightarrow x=\dfrac{37}{5},-\dfrac{43}{2},\dfrac{80}{3}$

高二：設$n\in\mathbb{N}$，且方程式$2x^4+2\sqrt{2}(n+1)x^3+(n^2+n+4)x^2+2\sqrt{2}(n+1)x+(n^2+n+2)=0$的兩實根分別為$\alpha_n$與$\beta_n$，試求$\dfrac{1}{{\alpha_5}^2+{\beta_5}^2}+\dfrac{1}{{\alpha_6}^2+{\beta_6}^2}+\cdots+\dfrac{1}{{\alpha_{15}}^2+{\beta_{15}}^2}$的值為何？

【解】$2x^4+2\sqrt{2}(n+1)x^3+(n^2+n+4)x^2+2\sqrt{2}(n+1)x+(n^2+n+2)$

$=(x^2+1)[2x^2+2\sqrt{2}(n+1)x+(n^2+n+2)]=0$

又$[2\sqrt{2}(n+1)]^2-4\times2\times(n^2+n+2)=8(n-1)\geq0$

故實數α_n與β_n為$2x^2+2\sqrt{2}(n+1)x+(n^2+n+2)=0$的兩根

因此${\alpha_n}^2+{\beta_n}^2=(\alpha_n+\beta_n)^2-2\alpha_n\beta_n=[\sqrt{2}(n+1)]^2-(n^2+n+2)=n(n+3)$

$$\sum_{n=5}^{15}\frac{1}{{\alpha_n}^2+{\beta_n}^2}=\sum_{n=5}^{15}\frac{1}{n(n+3)}=\frac{1}{3}\sum_{n=5}^{15}[\frac{1}{n}-\frac{1}{n+3}]$$

$$=\frac{1}{3}[(\frac{1}{5}-\frac{1}{8})+(\frac{1}{6}-\frac{1}{9})+(\frac{1}{7}-\frac{1}{10})+(\frac{1}{8}-\frac{1}{11})+\cdots+(\frac{1}{13}-\frac{1}{16})+(\frac{1}{14}-\frac{1}{17})+(\frac{1}{15}-\frac{1}{18})]$$

$$=\frac{1}{3}(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{1}{16}-\frac{1}{17}-\frac{1}{18})=\frac{1}{3}(\frac{107}{210}-\frac{433}{2448})=\frac{1}{3}\times\frac{28501}{85680}=\frac{28501}{257040}$$